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HOW OFTEN IS A POLYGON BOUNDED BY THREE SIDES?

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ABSTRACT

Let L_n be the set of lines (no two parallel) determining an n-sided bounded face F in the Euclidean plane. We show that the number, $f(L_n)$, of triples from L_n that determine a triangle containing F satisfies

$$n-2 \leqq f(L_n) \leqq \frac{n}{6}\left[\frac{n^2-1}{4}\right]$$

and these bounds are best. This result is generalized to d-dimensional Euclidean space (without the claim that the upper bound is attainable).

1. Introduction and notation

Let L_n be a set of $n \geqq 3$ lines in general position in the Euclidean plane such that some bounded face F determined by L_n is n-sided. We let $f(L_n)$ be the number of triples of lines of L_n that form a triangle containing F and will prove the following result in the third section:

THEOREM 1.

$$n-2 \leqq f(L_n) \leqq \frac{n}{6}\left[\frac{n^2-1}{4}\right]$$

and these bounds are best.

Our proof of this result will involve the consideration of certain properties of a set P of points p_i, $1 \leqq i \leqq m$, in general position on a circle S (no two antipodal). Each pair of distinct points p_i and p_j determine two arcs of S: the "arc p_ip_j" will mean the smaller of these and we refer to the set of all such smaller arcs as "the arcs of P". A triple of points of P not contained in any semicircle of S will be called a *central triple* (since the triangle formed by these points contains the center of S). The point of S antipodal to p_i will be denoted by p'_i, $1 \leqq i \leqq n$. We note that a triple $\{p_i, p_j, p_k\}$ is central iff the arc p_jp_k contains p'_i.

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2. Some properties of points on a circle

LEMMA 1. *Let P be a set of points p_i, $1 \leqq i \leqq n$, in general position on a circle S.*
(i) *The union of the arcs of P is S or is contained in a semicircle of S.*
(ii) *If one point of P is a member of a central triple then each point is.*
(iii) *If every three points of P are contained in a semicircle of S then they all are.*

PROOF. (i) If the union of the arcs of P is not S it is an arc C with endpoints p_i and p_j, say. The arc p_ip_j is either $S - C$ or C; in either case we are done.

(ii) If some point p_i is not a member of any central triple then the antipodal point p_i' cannot lie on any arc of P and so by (i) all points of P lie in a semicircle. But then no point of P is a member of a central triple.

(iii) The point p_i is not a member of a central triple so that p_i' does not lie in any arc of P. By (i) some semicircle of S contains all points of P.

LEMMA 2. *Let P_n be a set of points p_i, $1 \leqq i \leqq n$, $n \geqq 3$, in general position on a circle S and let $g(P_n)$ be the number of central triples of members of P_n. Then, if $g(P_n) > 0$, we have*

$$n - 2 \leqq g(P_n) \leqq \frac{n}{6}\left[\frac{n^2-1}{4}\right]$$

and these bounds are best.

PROOF. Let x_i and y_i be the numbers of points $p_j \neq p_i$ in the two semicircles of S with endpoint p_i. Then $x_i + y_i = n - 1$ and the number of triples not central is

$$\binom{n}{3} - g(P_n) = \frac{1}{2}\sum_{i=1}^{n}\left[\binom{x_i}{2} + \binom{y_i}{2}\right] = -\frac{1}{2}\binom{n}{2} + \frac{1}{4}\sum_{i=1}^{n}(x_i^2 + y_i^2)$$

so that

$$g(P_n) \leqq \binom{n}{3} + \frac{1}{2}\binom{n}{2} - \frac{1}{4}\begin{cases} 2n\left(\dfrac{n-1}{2}\right)^2, & n \text{ odd} \\ n\left(\dfrac{n^2}{4} + \dfrac{(n-2)^2}{4}\right), & n \text{ even} \end{cases}$$

$$= \left.\begin{cases} \dfrac{n(n^2-1)}{24}, & n \text{ odd} \\ \dfrac{n(n^2-4)}{24}, & n \text{ even} \end{cases}\right\} = \frac{n}{6}\left[\frac{n^2-1}{4}\right].$$

We assume $g(P_n)>0$ and prove $g(P_n)\geqq n-2$ by induction, it being obvious when $n=3$. For $n\geqq 4$, since P_n contains a central triple, for each $p_i\in P_n$ not in that triple, $P_n-\{p_i\}$ contains the same central triple so that by the inductive hypothesis it contains at least $n-3$ central triples. By (ii) of Lemma 1, p_i is also in a central triple so that $g(P_n)\geqq n-2$.

The lower bound may be realized by grouping p_i close to p_1 for $2\leqq i\leqq n-2$ and placing p_{n-1} and p_n near the ends of a diameter normal to the diameter through p_i. The upper bound may be realized by spacing n points nearly equally about a circle.

3. Proof of Theorem 1

Let S be any circle in the interior of F and let 0 be its center. Let P_n be the set of points of intersection p_i, $1\leqq i\leqq n$, of S and the rays from 0 normal to the lines l_i of L_n. As F is bounded the points p_i cannot all lie in some semicircle of S since the ray from 0 through the point antipodal to the center of such an arc would not intersect any of the lines of L_n. Consequently $f(L_n)>0$ and we see that if $\{p_i,p_j,p_k\}$ is a triple of points of P_n corresponding to a triple of lines of L_n that form a triangle containing F then $\{p_i,p_j,p_k\}$ is a central triple. Conversely, if $\{p_i,p_j,p_k\}$ is a central triple then any ray from 0 intersects at least one of the arcs p_ip_j, p_jp_k and p_kp_i since, by Lemma 1, their union is S. But a ray intersecting arc p_ip_j, for example, also intersects at least one of the lines l_i and l_j. Consequently the triangle formed by l_i, l_j and l_k contains 0 and so F. We conclude that $f(L_n)=g(P_n)>0$ so that Theorem 1 is a consequence of Lemma 2. The bounds of Theorem 1 are attained by positioning lines tangent to a circle so as to obtain the configurations of points p_i described in the last paragraph of the last section.

4. Generalization to higher dimensions

With appropriate modifications the results of the last two sections generalize to d-dimensional Euclidean space E^d, $d\geqq 2$. We now let L_n be a set of $n\geqq d+1$ hyperplanes in general position in E^d such that some d-dimensional cell F determined by L_n is n-faced. We let $f_d(L_n)$ be the number of $(d+1)$-tuples of hyperplanes of L_n that bound a cell containing F.

Let S be a d-dimensional hypersphere in the interior of F and, as before, let p_i, $1\leqq i\leqq n$, be the intersection of the ray from the center 0 of S normal to the hyperplane l_i of L_n. Each set P' of d of the points p_i determines a hyperplane and this hyperplane cuts S into two caps the smaller of which is said to be the *cap* *of* S determined by P'. A set of $d+1$ of the points of $P = \{p_i \mid 1 \leqq i \leqq n\}$ will be called a *central* $(d+1)$-*tuple* iff they are not contained in any hemihypersphere of S.

Lemma 1 and its proof generalize, if the obvious changes are made. In particular in (ii) and (iii) we now speak of $(d+1)$-tuples in place of triples.

To generalize Lemma 2 we let $g_d(P_n)$ be the number of central $(d+1)$-tuples of members of P_n and for each set P'_j, $1 \leqq j \leqq \binom{n}{d-1}$ of $d-1$ points of P_n we let x_j and y_j be the numbers of points in the hemihyperspheres of S determined by the hyperplane containing 0 and all points of P'_j. We count only those points not in P'_j so that $x_j + y_j = n - d + 1$. Then the number of $(d+1)$-tuples not central is

$$\binom{n}{d+1} - g_d(P_n) \geqq \frac{1}{[(d+1)^2/4]} \sum_{j=1}^{k} \left[\binom{x_j}{2} + \binom{y_j}{2} \right] \tag{1}$$

since, as we will show, each such $(d+1)$-tuple is counted at most $[(d+1)^2/4]$ times.

A $(d+1)$-tuple of points of P_n is counted as a non-central $(d+1)$-tuple once for each pair of points of the $(d+1)$-tuple that lie on the same side of the hyperplane determined by 0 and the remaining $d-1$ points. This is equivalent to counting the faces of the convex hull of a set of $d+1$ points in E^{d-1} as may be seen, e.g., by projecting the hemihypersphere onto a hyperplane through its equator. It is known [1, pp. 169, 175] that the number of such faces is at most

$$f_{d-2}(d+1, d-1) = \begin{cases} \dfrac{d+1}{m}\dbinom{d-m}{m-1} & \text{for } d-1 = 2m \\[2ex] 2\dbinom{d-m}{m} & \text{for } d-1 = 2m+1 \end{cases} = [(d+1)^2/4].$$

From (1) we obtain

$$g_d(P_n) \leqq \binom{n}{d+1} - \frac{1}{2[(d+1)^2/4]} \left\{ \sum_{j=1}^{\binom{n}{d-1}} (x_j^2 + y_j^2) - \binom{n}{d-1}(n-d+1) \right\}$$

$$\leqq \binom{n}{d+1} + \binom{n}{d-1} \frac{(n-d+1)}{2[(d+1)^2/4]} - \frac{1}{2[(d+1)^2/4]} \cdot$$

$$\cdot \begin{cases} 2\dbinom{n}{d-1}\dfrac{(n-d+1)^2}{4}, & n-d \text{ odd} \\[2ex] \dbinom{n}{d-1}\left(\dfrac{(n-d)^2}{4} + \dfrac{(n-d+2)^2}{4}\right), & n-d \text{ even} \end{cases}$$

$$= \binom{n}{d+1} - \binom{n}{d-1} \begin{cases} \dfrac{(n-d)^2-1}{4[(d+1)^2/4]}, & n-d \text{ odd} \\[2ex] \dfrac{(n-d)^2}{4[(d+1)^2/4]}, & n-d \text{ even} \end{cases}$$

$$= \binom{n}{d+1} \begin{cases} \dfrac{n+1}{(d+1)(n-d+1)}, & n \text{ odd}, d \text{ odd} \\[2ex] \dfrac{n+1}{(d+1)(n-d)}, & n \text{ odd}, d \text{ even} \\[2ex] \dfrac{n}{(d+1)(n-d)}, & n \text{ even}, d \text{ odd} \\[2ex] \dfrac{n+2}{(d+2)(n-d+1)}, & n \text{ even}, d \text{ even} \end{cases}$$

The proof that $g_d(P_n) \geqq n-d$ and that this bound is attained for $d \geqq 3$ parallels that for the case $d=2$ and is omitted.

If we denote by $f_d(L_n)$ the function analogous to $f(L_n)$ in E^d we argue as before that $f_d(L_n) = g_d(P_n)$.

Configurations of n hyperplanes in E^d for which $f_d(L_n) = n-d$ can be constructed as described for the case $d=2$. However, we don't know whether the upper bound for $f_d(L_n)$ implied by (1) can be improved and we leave this as an open problem.

Reference

1. B. Grünbaum, *Convex Polytopes*, Interscience, New York, 1967.

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